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The Odd Exponentiated Half-Logistic-G Family: Properties, Characterizations and Applications

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Abstract

We introduce a new class of continuous distributions called the *odd exponentiated half-logistic-G family*. Some special models of the new family are provided. These special models are capable of modeling various shapes of aging and failure criteria. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics, probability weighted moments and characterizations are obtained. The maximum likelihood method is used for estimating the model parameters. The flexibility of the generated family is illustrated by means of three applications to real data sets.

Keywords: Generating Function · Maximum Likelihood · Order Statistic · T-X Family · Characterization.

Mathematics Subject Classification: 62E10 · 60E05.

1. Introduction

In many practical situations, classical distributions do not provide adequate fits to real data. Therefore, there has been an increased interest in developing more flexible distributions through extending the classical distributions via introducing additional shape parameters to the baseline model. Many generalized families of distributions have been proposed and studied over the last two decades for modeling data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences and finance. Some well-known families are the Marshall-Olkin-G by Marshall and Olkin (1997), the beta-G by Eugene *et al.* (2002), odd log-logistic-G by Gleaton and Lynch (2004, 2006), the transmuted-G by Shaw and Buckley (2009), the gamma-G by Zografos and Balakrishnan (2009), the Kumaraswamy-G by Cordeiro and de Castro (2011), the logistic-G by Torabi and Montazeri (2014), exponentiated generalized-G by Cordeiro *et al.* (2013), the McDonald-G by Alexander *et al.* (2012), T-X family by Alzaatreh *et al.* (2013), the Weibull-G by Bourguignon *et al.* (2014), the exponentiated half-logistic generated family by Cordeiro *et al.* (2014), the beta odd log-logistic generalized by Cordeiro *et al.* (2015), the generalized transmuted-G by Nofal *et al.* (2017), the Kumaraswamy transmuted-G by Afify

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et al. (2016a), the beta transmuted-H by Afify *et al.* (2016b), generalized odd log-logistic-G by Cordeiro *al.* (2017). Several mathematical properties of the extended distributions may be easily explored using mixture forms of exponentiated-G (exp-G) distributions.

One of the probability distributions which is a member of the family of the logistic distribution is the half logistic distribution. The probability density function (pdf) and cumulative density function (cdf) of the half-logistic (HL) distribution are, respectively, given by

$$f\left(x\right)=\frac{2\,\lambda\,\exp\left(-\lambda x\right)}{\left(1+\exp\left(-\lambda x\right)\right)^{2}},$$

$$F\left(x\right)=\frac{1-\exp\left(-\lambda x\right)}{1+\exp\left(-\lambda x\right)},$$

where $x>0$ and $\lambda>0$ is a shape parameter.

The HL distribution has not receive much attention from researchers in terms of generalization. Furthermore, the density function of HL distribution has unimodal or reversed J-shaped and this property is a disadvantage of the HL distribution since the empirical approaches to real data are often non-monotone hazard rate function shapes such as unimodal, bathtub and various shaped, specifically in the lifetime applications. Hence, in this research work, we present a new generalization of the HL distribution.

The goal of this study is to propose a new flexible family of distributions called the *odd exponentiated half logistic-G* (OEHL-G for short) family of distributions using the HL distribution as the generator and study its mathematical properties. This way, we will utilize the flexibility of the baseline distribution for modelling the data.

The cdf and pdf of the OEHL-G family are given, respectively, by

$$F(x;\alpha,\lambda,\boldsymbol{\xi})=\left\{\frac{1-\exp\left[\frac{-\lambda\;G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}{1+\exp\left[\frac{-\lambda\;G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}\right\}^{\alpha},\quad x\in\mathrm{R},\tag{1}$$

and

$$f(x;\alpha,\lambda,\boldsymbol{\xi})=\frac{2\alpha\lambda\,g(x,\boldsymbol{\xi})\exp\left[\frac{-\lambda\;G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\left\{1-\exp\left[\frac{-\lambda\;G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}^{\alpha-1}}{\bar{G}(x;\boldsymbol{\xi})^{2}\left\{1+\exp\left[\frac{-\lambda\;G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}^{\alpha+1}},\quad x\in\mathrm{R},\tag{2}$$

where $g(x;\boldsymbol{\xi})=dG(x;\boldsymbol{\xi})/dx$, α and λ are positive shape parameters and $\boldsymbol{\xi}$ is the vector of parameters for the baseline cdf G. Henceforth, a random variable with density (2) is denoted by $X\sim$OEHL-$G(\alpha,\lambda,\boldsymbol{\xi})$.

The reliability function, hazard rate function (hrf) and cumulative hazard rate function (chrf) of X are, respectively, given by

$$R(x;\alpha,\lambda,\boldsymbol{\xi})=1-\left\{\frac{1-\exp\left[\frac{-\lambda\,G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}{1+\exp\left[\frac{-\lambda\,G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}\right\}^{\alpha},$$

$$h(x;\alpha,\lambda,\boldsymbol{\xi})=\frac{2\alpha\lambda\, g(x,\boldsymbol{\xi})\bar{G}(x;\boldsymbol{\xi})^{-2}\left\{1+\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}^{-1}\left\{1-\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}^{\alpha-1}}{\exp\left[\frac{\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\left(\left\{1+\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}^{\alpha}-\left\{1-\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]\right\}\right)} \tag{3}$$

and

$$H(x;\alpha,\lambda,\boldsymbol{\xi})=-\log\left(1-\left\{\frac{1-\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}{1+\exp\left[\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]}\right\}^{\alpha}\right).$$

An interpretation of the OEHL-G family can be given as follows: Let T be a random variable with cdf $G(\cdot)$ describing a stochastic system. Let the random variable X represent the odds, the risk that the system following the lifetime T will be not working at time x is given by $G(x;\boldsymbol{\xi})/[1-G(x;\boldsymbol{\xi})]$. If we are interested in modeling the randomness of the odds by the exponentiated half-logistic cdf $\Pi(t)=\left[\frac{1-\mathrm{e}^{-\lambda t}}{1+\mathrm{e}^{-\lambda t}}\right]^{\alpha}$ (for $t>0$), the cdf of X is given by

$$Pr(X\le x)=\Pi\left(\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right)=\left[\frac{1-\mathrm{e}^{\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}}}{1+\mathrm{e}^{\frac{-\lambda\, G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}}}\right]^{\alpha}.$$

Furthermore, the basic motivations for using the OEHL-G family in practice are the following:

(1) to make the kurtosis more flexible compared to the baseline model;
(2) to produce a skewness for symmetrical distributions;
(3) to construct heavy-tailed distributions that are not longer-tailed for modeling real data;
(4) to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped;
(5) to define special models with all types of the hrfs;
(6) to provide consistently better fits than other generated models under the same baseline distribution.

The rest of the paper is organized as follows. In Section 2, we give a very useful linear representation for the density function of the family. In Section 3, we present three special models and plots of their pdfs and hrfs. In Section 4, we obtain some of its general mathematical properties including asymptotics, ordinary and incomplete moments, skewness and kurtosis, quantile and generating functions, quantile power series, entropies, order statistics and probability weighted moments (PWMs). Section 5 is devoted to characterizations of the OEHL-G family. The maximum likelihood estimation of the model parameters is addressed in Section 6. In Section 7, we provide two applications to real data to illustrate the flexibility of the new family. Simulation results to assess the performance of the maximum likelihood estimation method are reported in Section 8. Finally, we give some concluding remarks in Section 9.

2. Linear representation

In this section, we provide a useful representation for the OEHL-G pdf. The pdf (2) can be expressed as

$$f(x)=\frac{2\alpha\lambda g(x)}{\bar{G}(x)^2}e^{\frac{-\lambda\ G(x)}{\bar{G}(x)}}\underbrace{\left\{1-\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{\alpha-1}}_{A}\underbrace{\left\{1+\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{-\alpha-1}}_{B}. \quad (4)$$

Using the generalized binomial series, we have

$$A=\sum_{j=0}^{\infty}(-1)^j\binom{\alpha-1}{j}\exp\left[\frac{-\lambda jG(x)}{\bar{G}(x)}\right] \quad \text{and} \quad B=\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\exp\left[\frac{-\lambda iG(x)}{\bar{G}(x)}\right]. \quad (5)$$

Combining equations (4) and (5), we obtain

$$f(x)=\frac{2\alpha\lambda g(x)}{\bar{G}(x)^2}\sum_{j,i=0}^{\infty}(-1)^j\binom{-\alpha-1}{i}\binom{\alpha-1}{j}\exp\left[\frac{-\lambda\,(j+i+1)\,G(x)}{\bar{G}(x)}\right].$$

From the exponential series, we can write

$$f(x)=2\alpha\lambda g(x)\sum_{j,i,k=0}^{\infty}\frac{(-1)^{j+k}}{k!}\left[\lambda\,(j+i+1)\right]^k\binom{-\alpha-1}{i}\binom{\alpha-1}{j}G(x)^k\bar{G}(x)^{-k-2}. \quad (6)$$

Consider the power series

$$(1-z)^{-q}=\sum_{k=0}^{\infty}(-1)^k\binom{-q}{k}z^k. \quad (7)$$

Using the power series in (7), equation (6) can be expressed as

$$f(x)=2\alpha\lambda g(x)\textstyle\sum_{j,i,k,l=0}^{\infty}\frac{(-1)^{j+k+l}}{k!}\left[\lambda\,(j+i+1)\right]^k\binom{\alpha-1}{j}\times\binom{-\alpha-1}{i}\binom{-k-2}{l}G(x)^{k+l},$$

or, equivalently, we can write

$$f(x)=\sum_{k,l=0}^{\infty}a_{k,l}\ h_{k+l+1}\left(x\right), \quad (8)$$

where

$$a_{k,l}=2\alpha\lambda\sum_{j,i=0}^{\infty}\frac{(-1)^{j+k+l}\left[\lambda\,(j+i+1)\right]^k}{k!\,(k+l+1)}\binom{-\alpha-1}{i}\binom{\alpha-1}{j}\binom{-k-2}{l}$$

and $h_{k+l+1}\left(x\right)=(k+l+1)\,g(x)G(x)^{k+l}$ is the Exp-G density with power parameter $(k+l+1)$. Thus, several mathematical properties of the OEHL-G family can be obtained simply from those properties of the Exp-G family.

The cdf of the OEHL-G family can also be expressed as a mixture of Exp-G cdfs. By integrating (8), we obtain the same mixture representation

$$PrimaryXXX \cdot \ SecondaryXXXF(x) = \sum_{k,l=0}^{\infty} a_{k,l} \ H_{k+l+1}\left(x\right),$$

where $H_{k+l+1}\left(x\right)$ is the cdf of the Exp-G family with power parameter $(k+l+1)$.

3. Special models

In this section, we provide three special models of the OEHL-G family. The pdf (2) will be most tractable when the cdf $G(x;\boldsymbol{\xi})$ and the pdf $g(x;\boldsymbol{\xi})$ have simple analytic expressions.

3.1 The OEHL-Weibull (OEHL-W) distribution

By taking $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ in (2) to be the cdf and pdf of the Weibull (W) distribution with cdf and pdf $G(x;a,b) = 1 - e^{-(x/b)^a}$ and $g(x;a,b) = ab^{-a}x^{a-1}e^{-(x/b)^a}$, respectively, where $a > 0$ is a shape parameter and $b > 0$ is a scale parameter. The pdf of the OEHL-W reduces (for $x > 0$) to

$$f(x) = \frac{2\alpha\lambda ab^{-a}x^{a-1}\exp\left[(x/b)^a\right]\left(1-\exp\left\{-\lambda\left[e^{(x/b)^a}-1\right]\right\}\right)^{\alpha-1}}{\exp\left\{-\lambda\left[e^{(x/b)^a}-1\right]\right\}\left(1+\exp\left\{-\lambda\left[e^{(x/b)^a}-1\right]\right\}\right)^{\alpha+1}}.$$

For $b = 1$, we have the OEHL-exponential (OEHL-E) distribution and for $b = 2$, we obtain the OEHL-Rayleigh (OEHL-R) distribution. Plots of the pdf and hrf of OEHL-W distribution for selected parameter values are shown in Figure 1. Figure 1 demonstrates that the OEHL-W ensures rich shaped distributions with various shapes for modeling. It brings unimodal, bathtub and decreasing shape properties. Figure 1 also reveals that this family can produce flexible hrf shapes such as decreasing, increasing, bathtub, upside-down bathtub, firstly unimodal. Other shapes can be obtained using another distribution. These shape properties show that the OEHL-W family can be very useful to fit different data sets with various shapes.

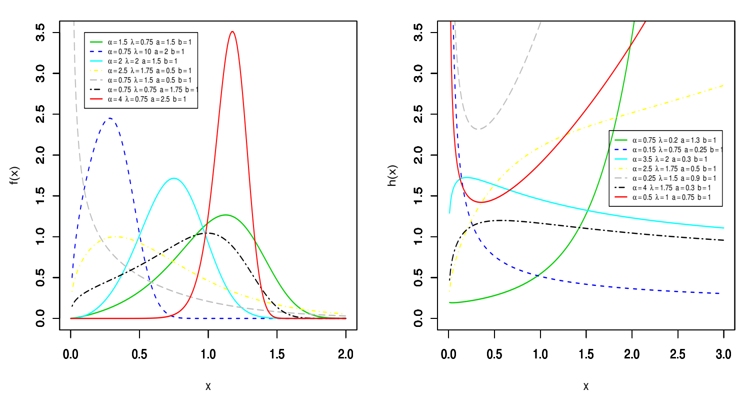

Figure 1. Plots of the pdf and hrf of the OEHL-W distribution for the selected parameter values.

3.2 The OEHL-gamma (OEHL-Ga) distribution

By taking $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ in (2) to be the cdf $G(x;a,b)=\gamma\left(a,x/b\right)/\Gamma\left(a\right)$ and the pdf $g(x;a,b)=x^{a-1}\mathrm{e}^{-x/b}/b^{a}\Gamma\left(a\right)$ of the gamma distribution, where $a>0$ is a shape parameter and $b>0$ is a scale parameter, the pdf of the OEHL-Ga (for $x>0$) reduces to

$$f(x)=\frac{2\alpha\lambda x^{a-1}\exp\left(-x/b\right)\exp\left[\frac{-\lambda\gamma(a,x/b)/\Gamma(a)}{1-\gamma(a,x/b)/\Gamma(a)}\right]\left\{1-\exp\left[\frac{-\lambda\gamma(a,x/b)/\Gamma(a)}{1-\gamma(a,x/b)/\Gamma(a)}\right]\right\}^{\alpha-1}}{b^{a}\Gamma\left(a\right)\left[1-\frac{-\lambda\gamma(a,x/b)/\Gamma(a)}{1-\gamma(a,x/b)/\Gamma(a)}\right]^{2}\left\{1+\exp\left[\frac{-\lambda\gamma(a,x/b)/\Gamma(a)}{1-\gamma(a,x/b)/\Gamma(a)}\right]\right\}^{\alpha+1}}.$$

Plots of the pdf and hrf of the OEHL-Ga distribution for selected parameter values are shown in Figure 2. Figure 2 shows that the pdf of the OEHL-Ga is right-skewed and nearly symmetric. As seen in Figure 2, a characteristic of the OEHL-Ga distribution is that its hrf can be monotonically increasing or bathtub depending basically on the parameter values.

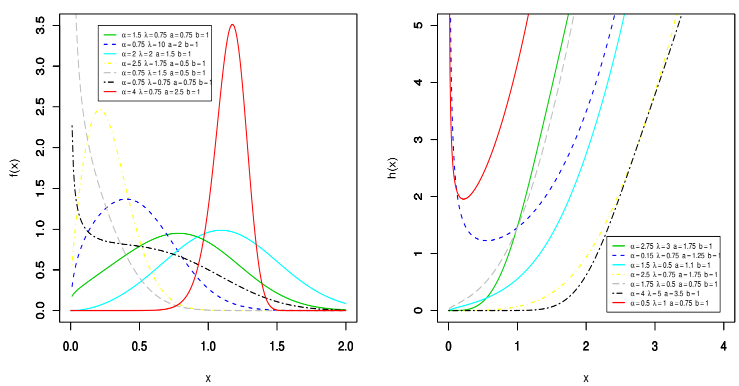

Figure 2. Plots of the pdf and hrf of the OEHL-Ga distribution for the selected parameter values.

3.3 The OEHL-normal (OEHL-N) distribution

The OEHL-N distribution is defined from (2) by taking $G(x;\mu,\sigma^{2})=\Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g(x;\mu,\sigma^{2})=\sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$ for the cdf and pdf of the normal (N) distribution with a location parameter $\mu\in\mathrm{R}$ and a scale positive parameter σ^{2}, where $\Phi\left(.\right)$ and $\phi\left(.\right)$ are the pdf and cdf of the standard N distribution, respectively. The OEHL-N pdf is given (for $x\in\mathrm{R}$) by

$$f(x)=\frac{2\alpha\lambda\phi\left(\frac{x-\mu}{\sigma}\right)\exp\left[\frac{-\lambda\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]\left\{1-\exp\left[\frac{-\lambda\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]\right\}^{\alpha-1}}{\sigma\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{2}\left\{1+\exp\left[\frac{-\lambda\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]\right\}^{\alpha+1}}.$$

For $\mu=0$ and $\sigma=1$, we obtain the standard OEHL-N distribution. Plots of the OEHL-N density for selected parameter values are shown in Figure 3. Figure 3 reveals that the density function of OEHL-N has left-skewed, nearly symmetric and non-monotonic shapes.

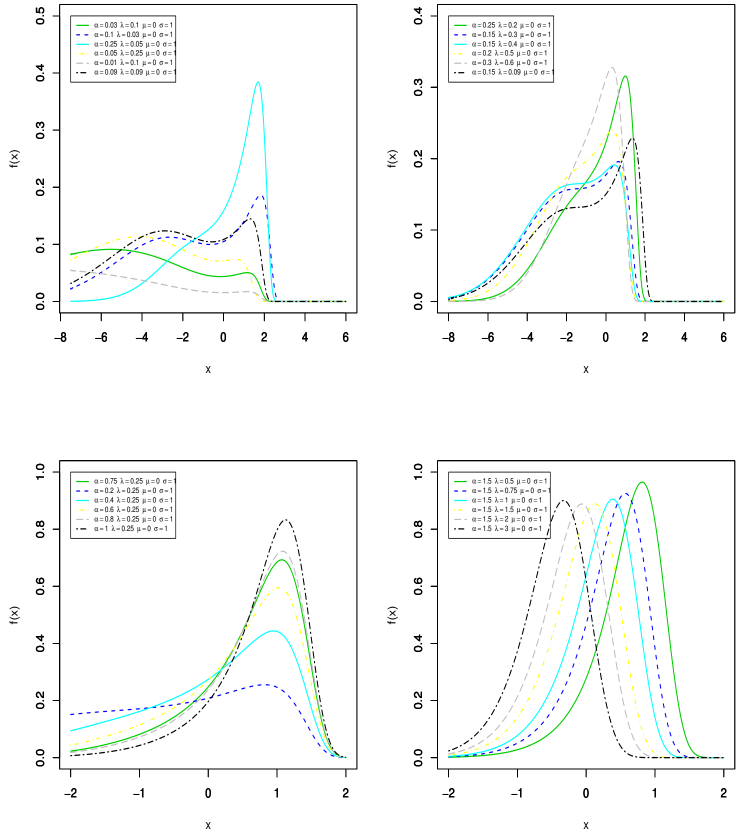

Figure 3. Plots of the density function of the OEHL-N distribution for the selected parameter values.

4. Properties

In this section, we obtain some general properties of the OEHL-G family including the ordinary and incomplete moments, generating function, entropies, order statistics and probability weighted moments (PWMs).

4.1 Asymptotics

Let $a = \inf\{x|G(x) > 0\}$, then, the asymptotics of equations (1), (2) and (3) as $x \to a$ are given by

$$F(x) \sim \frac{\lambda^{\alpha}}{2^{\alpha}} G(x)^{\alpha} \qquad \text{as } x \to a,$$

$$f(x) \sim \frac{\alpha\lambda^{\alpha}}{2^{\alpha}} g(x)G(x)^{\alpha-1} \qquad \text{as } x \to a,$$

$$h(x) \sim \frac{\alpha\lambda^{\alpha}}{2^{\alpha}} g(x)G(x)^{\alpha-1} \qquad \text{as } x \to a.$$

The asymptotics of equations (1), (2) and (3) as $x \to \infty$ are given by

$$1 - F(x) \sim 2\alpha e^{\frac{-\lambda}{\bar{G}(x)}} \qquad \text{as } x \to \infty,$$

$$f(x) \sim \frac{2\alpha\lambda g(x) e^{\frac{-\lambda}{\bar{G}(x)}}}{\bar{G}(x)^2} \qquad \text{as } x \to \infty,$$

$$h(x) \sim \frac{2\alpha\lambda g(x)}{\bar{G}(x)^2} \qquad \text{as } x \to \infty.$$

4.2 Ordinary and incomplete moments

Henceforth, T_{k+l+1} denotes the Exp-G random variable with power parameter $k+l+1$. The rth moment of X, say μ_r', follows from (8) as

$$\mu_r' = E(X^r) = \sum_{k,l=0}^{\infty} a_{k,l}\; E\left(T_{k+l+1}^r\right). \tag{9}$$

The nth central moment of X is given by

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} \left(-\mu_1'\right)^{n-r} E(X^r) = \sum_{r=0}^{n}\sum_{k,l=0}^{\infty} a_{k,l} \binom{n}{r} \left(-\mu_1'\right)^{n-r} E\left(T_{k+l+1}^r\right).$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu_n' - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r\, \mu_{n-r}',$$

where $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + \mu_1'^3$, etc. The measures of skewness and kurtosis can be calculated from the ordinary moments using well-known relationships.

The sth incomplete moment of X can be expressed from (8) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, \mathrm{d}x = \sum_{k,l=0}^{\infty} a_{k,l} \int_{-\infty}^{t} x^s\, h_{k+l+1}(x)\, \mathrm{d}x. \tag{10}$$

$\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/(\pi\mu_1')$ and $L(\pi) = \varphi_1(q)/\mu_1'$, respectively, where μ_1' given by (9) with $r=1$ and $q = Q(\pi)$ is the qf of X at π. These curves are very useful in economics, reliability, demography, insurance and medicine.

Now, we provide two ways to determine $\varphi_1(t)$. First, a general equation for $\varphi_1(t)$ can be derived from (10) as

$$\varphi_1(t) = \sum_{k,l=0}^{\infty} a_{k,l}\; J_{k+l+1}(t),$$

where $J_{k+l+1}(t) = \int_{-\infty}^{t} x\, h_{k+l+1}(x)\, dx$ is the first incomplete moment of the Exp-G distribution.

A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k,l=0}^{\infty} a_{k,l}\, v_{k+l+1}(t)\,,$$

where $v_{k+l+1}(t) = (k+l+1)\int_0^{G(t)} Q_G(u)\, u^{k+l} du$ which can be computed numerically and $Q_G(u)$ is the qf corresponding to $G(x;\boldsymbol{\xi})$, i.e., $Q_G(u) = G^{-1}(u;\boldsymbol{\xi})$.

Figures 4 and 5 display the mean and variance measures of OEHL-N and OEHL-W distributions. Based on these figures, we conclude that: when parameter α increases, mean increases and variance decreases; when parameter λ increases, mean and variance decrease.

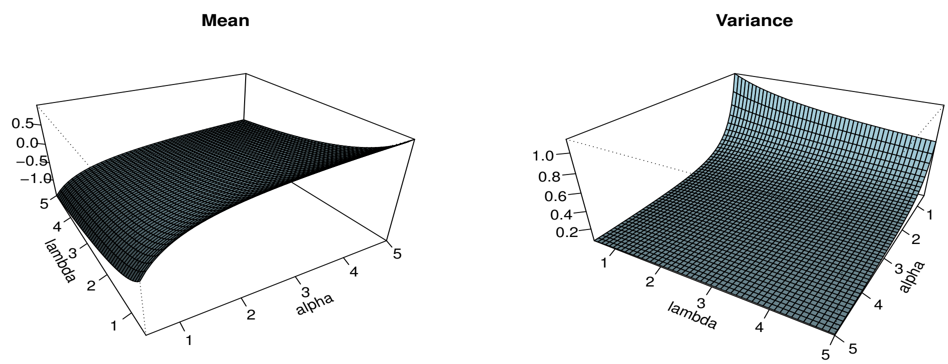

Figure 4. Plots of the mean and variance of the OEHL-N distribution for $\mu = 0$, $\sigma = 1$.

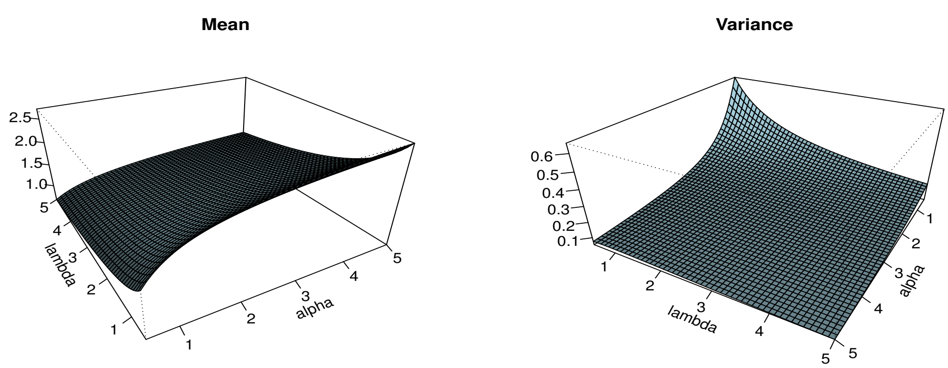

Figure 5. Plots of the mean and variance of the OEHL-W distribution for $a = 2$, $b = 2$.

4.3 Quantile and generating functions

The quantile function (qf) of the OEHL-G distribution follows, by inverting (1), as

$$X_U = Q_G\left[\frac{-\log(1-u^{\frac{1}{\alpha}}) + \log(1+u^{\frac{1}{\alpha}})}{\lambda - \log(1-u^{\frac{1}{\alpha}}) + \log(1+u^{\frac{1}{\alpha}})}\right], \tag{11}$$

where $Q_G(.)$ denote the qf of X and $u \sim U\,(0,1)$.

Here, we provide two formulae for the moment generating function (mgf) $M_X\,(t) = E\left(\mathrm{e}^{t\,X}\right)$ of X which can be derived from equation (8).The first one is given by

$$M_X\,(t) = \sum_{k,l=0}^{\infty} a_{k,l}\, M_{k+l+1}\,(t)\,,$$

where $M_{k+l+1}\,(t)$ is the mgf of T_{k+l+1}. Hence, $M_X\,(t)$ can be determined from the Exp-G generating function.

The second formula for $M_X\,(t)$ can be expressed as

$$M_X\,(t) = \sum_{k,l=0}^{\infty} a_{k,l}\, \tau\,(t,k)\,,$$

where $\tau\,(t,k) = \int_0^1 \exp\left[t\, Q_G\,(u)\right]\, u^{k+l} du$. The skewness and kurtosis plots of the OEHL-Ga and OEHL-W are given in Figure 4. These plots indicate that the members of the this family can model various data types in terms of skewness and kurtosis.

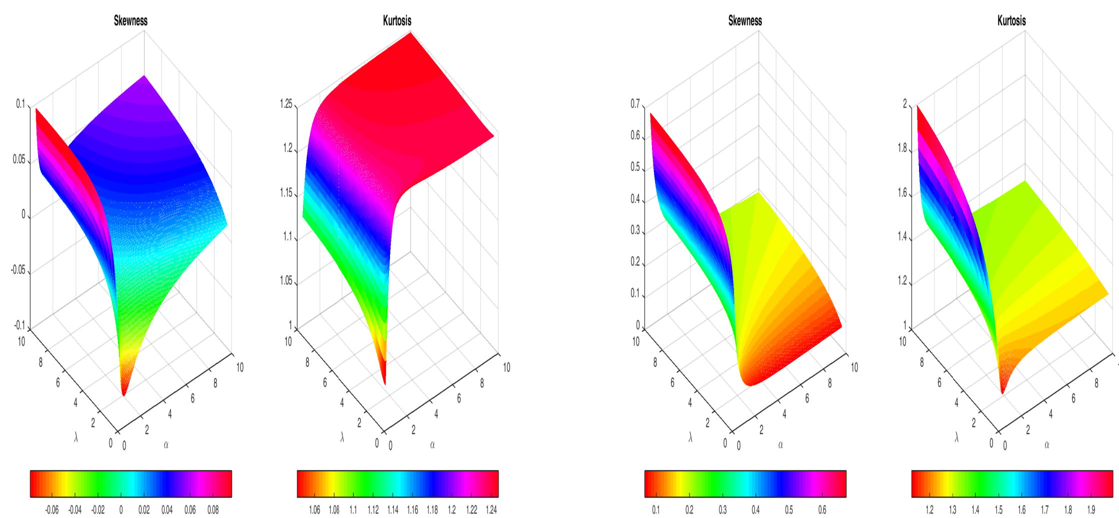

Figure 6. Plots of skewness and kurtosis of OEHL-Ga (left panel) and OEHL-W (right panel) distribution for several values of parameters.

4.4 Quantile power series

In this subsection, we derive a power series for the qf $x = Q(u) = F^{-1}(u)$ of X by expanding (11). If $Q_G(u)$ does not have a closed-form expression, it can expressed as a power series

$$Q_G(u) = \sum_{i=0}^{\infty} a_i\, u^i, \tag{12}$$

where the coefficients $a_i's$ are suitably chosen real numbers. They depend on the parameters of the baseline G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in equation (12). As a simple example, for the normal $N(0,1)$ distribution, $a_i = 0$ for $i = 0, 2, 4, \ldots$ and $a_1 = 1$, $a_3 = 1/6$, $a_5 = 7/120$ and $a_7 = 127/7560, \ldots$

Throughout the paper, we use a result of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{13}$$

where $c_{n,0} = a_0^n$ and the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}. \tag{14}$$

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in (11), namely

$$A = \frac{-\log(1-u^{\frac{1}{\alpha}}) + \log(1+u^{\frac{1}{\alpha}})}{\lambda - \log(1-u^{\frac{1}{\alpha}}) + \log(1+u^{\frac{1}{\alpha}})}.$$

First, we have

$$\begin{aligned} -\log(1-u^{\frac{1}{\alpha}}) &= \sum_{i=1}^{\infty} \frac{u^{\frac{i}{\alpha}}}{i} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^j\,(1-u)^j}{i}\binom{\frac{i}{\alpha}}{j} \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{j+k}}{i}\binom{\frac{i}{\alpha}}{j}\binom{j}{k}u^k = \sum_{k=0}^{\infty} a_k\, u^k, \end{aligned}$$

where $a_k = \sum\limits_{i=1}^{\infty}\sum\limits_{j=k}^{\infty} \frac{(-1)^{j+k}}{i}\binom{\frac{i}{\alpha}}{j}\binom{j}{k}$.

Also, we can write

$$\begin{aligned} \log(1+u^{\frac{1}{\alpha}}) &= -\sum_{i=1}^{\infty} \frac{(-1)^i\, u^{\frac{i}{\alpha}}}{i} = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j+1}\,(1-u)^j}{i}\binom{\frac{i}{\alpha}}{j} \\ &= \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j+k+1}}{i}\binom{\frac{i}{\alpha}}{j}\binom{j}{k}u^k = \sum_{k=0}^{\infty} b_k\, u^k, \end{aligned}$$

where $b_k = \sum\limits_{i=1}^{\infty}\sum\limits_{j=k}^{\infty} \frac{(-1)^{i+j+k+1}}{i}\binom{\frac{i}{\alpha}}{j}\binom{j}{k}$.

Then, using the ratio of two power series we can write

$$A = \frac{\sum_{k=0}^{\infty} \alpha_k \ u^k}{\sum_{k=0}^{\infty} \beta_k \ u^k} = \sum_{k=0}^{\infty} \delta_k \ u^k, \tag{15}$$

where $\alpha_k = a_k + b_k$ for $k \geq 0$, $\beta_0 = \lambda + \alpha_0$ and $\beta_k = \alpha_k$ for $k \geq 1$ and $\delta_0 = \alpha_0/\beta_0$ and for $k \geq 1$, we have

$$\delta_k = \frac{1}{\beta_0}\left[\alpha_k - \frac{1}{\beta_0}\sum_{r=1}^{k} \beta_r \delta_{k-r}\right].$$

Then, the qf of X can be expressed using (11) as

$$Q(u) = Q_G\left(\sum_{k=0}^{\infty} \delta_k \, u^k\right). \tag{16}$$

For any baseline G distribution, we combine (12) and (16) to obtain

$$Q(u) = Q_G\left(\sum_{m=0}^{\infty} \delta_m \, u^m\right) = \sum_{i=0}^{\infty} a_i \left(\sum_{m=0}^{\infty} \delta_m \, u^m\right)^i.$$

Then using (13) and (14), we have

$$Q(u) = \sum_{m=0}^{\infty} e_m \, u^m, \tag{17}$$

where $e_m = \sum_{i=0}^{\infty} a_i \, d_{i,m}$, and, for $i = 0, 1, \ldots$, $d_{i,0} = \delta_0^i$ and (for $m > 1$)

$$d_{i,m} = (m\,\delta_0)^{-1} \sum_{n=1}^{m} [n(i+1) - m]\, \delta_n \, d_{i,m-n}.$$

Equation (17) reveals that the qf of the OEHL-G family can be expressed as a power series. Then, several mathematical quantities of X can be reduced to integrals over $(0, 1)$ based on this power series.

Let $W(\cdot)$ be any integrable function on the real line. We can write

$$\int_{-\infty}^{\infty} W(x)\, f(x)\mathrm{d}x = \int_0^1 W\left(\sum_{m=0}^{\infty} e_m \, u^m\right) \mathrm{d}u. \tag{18}$$

Equations (17) and (18) are the main results of this section since we can obtain various OEHL-G mathematical properties based on them. In fact, they can follow by using the integral on the right-hand side for special $W(\cdot)$ functions, which are usually simpler than if they were based on the left-hand integral. For the great majority of these quantities, we can adopt twenty terms in this power series.

The formulae derived throughout the paper can be easily handled in most symbolic computation platforms such as Maple, Mathematica and Matlab.

4.5 Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is given by

$$I_\theta(X) = (1-\theta)^{-1} \log\left(\int_{-\infty}^{\infty} f(x)^\theta \, \mathrm{d}x\right), \qquad \theta > 0 \text{ and } \theta \neq 1.$$

Using the pdf (2), we can write

$$f(x)^\theta = \frac{(2\alpha\lambda)^\theta g(x)^\theta \exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\left\{1-\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{\theta(\alpha-1)}}{\bar{G}(x)^{2\theta}\left\{1+\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{\theta(\alpha+1)}}.$$

Applying the binomial series to the last term, the last equation reduces to

$$f(x)^\theta = \frac{(2\alpha\lambda)^\theta g(x)^\theta}{\bar{G}(x)^{2\theta}} \sum_{j,i=0}^{\infty} (-1)^j \binom{\theta(\alpha-1)}{j}\binom{-\theta(\alpha+1)}{i} \exp\left[\frac{-\lambda(j+i+1)G(x)}{\bar{G}(x)}\right].$$

Using the exponential series and then the power series (7) , we have

$$f(x)^\theta = \sum_{j,i,k,l=0}^{\infty} \frac{(-1)^{j+k+l}(2\alpha\lambda)^\theta g(x)^\theta}{k!\,[\lambda(j+i+1)]^{-k}} \binom{\theta(\alpha-1)}{j}\binom{-\theta(\alpha+1)}{i}\binom{-k-2\theta}{l} G(x)^{k+l}.$$

Then, the Rényi entropy of the OEHL-G class is given by

$$I_\theta(X) = (1-\theta)^{-1}\log\left[\sum_{k,l=0}^{\infty} \upsilon_{k,l}\int_{-\infty}^{\infty} g(x)^\theta G(x)^{k+l}\mathrm{d}x\right],$$

where

$$\upsilon_{k,j} = (2\alpha\lambda)^\theta \sum_{j,i=0}^{\infty} \frac{(-1)^{j+k+l}}{k!}[\lambda(j+i+1)]^k \binom{\theta(\alpha-1)}{j}\binom{-\theta(\alpha+1)}{i}\binom{-k-2\theta}{l}.$$

The θ-entropy can be obtained as

$$H_\theta(X) = (1-\theta)^{-1}\log\left[1-\sum_{k,l=0}^{\infty}\upsilon_{k,l}\int_{-\infty}^{\infty} g(x)^\theta G(x)^{k+l}\mathrm{d}x\right].$$

The Shannon entropy of a random variable X is a special case of the Rényi entropy when $\theta \uparrow 1$. The Shannon entropy, say SI, is defined by $SI = E\{-[\log f(X)]\}$, which follows by taking the limit of $I_\theta(X)$ as θ tends to 1.

4.6 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the OEHL-G family. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}\left(x\right)=\frac{f\left(x\right)}{\mathrm{B}\left(i,n-i+1\right)}\sum_{j=0}^{n-i}\left(-1\right)^{j}\binom{n-i}{j}F\left(x\right)^{j+i-1}, \tag{19}$$

where $B(\cdot,\cdot)$ is the beta function.

Using equations (1) and (2), we have

$$\begin{aligned} f\left(x\right)F\left(x\right)^{j+i-1} &= \frac{2\alpha\lambda g(x)}{\bar{G}(x)^{2}}\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\left\{1-\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{\alpha(j+i)-1} \\ &\times\left\{1+\exp\left[\frac{-\lambda G(x)}{\bar{G}(x)}\right]\right\}^{-\alpha(j+i)-1}. \end{aligned}$$

After applying the generalized binomial and exponential series, we obtain

$$\begin{aligned} f\left(x\right)F\left(x\right)^{j+i-1} &= \sum_{s,w,k,l=0}^{\infty}\frac{\left(-1\right)^{s+k+l}2\alpha\lambda g(x)}{k!\left[\lambda\left(s+w+1\right)\right]^{-k}}\binom{\alpha\left(j+i\right)-1}{s} \\ &\times\binom{-\alpha\left(j+i\right)-1}{w}\binom{-k-2}{l}G(x)^{k+l}. \end{aligned} \tag{20}$$

Substituting (20) in equation (19), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}\left(x\right)=\sum_{k,l=0}^{\infty}b_{k,l}\;h_{k+l+1}\left(x\right),$$

where $h_{k+l+1}\left(x\right)$ is the Exp-G density with power parameter $(k+l+1)$ and

$$\begin{aligned} b_{k,l} &= \sum_{j=0}^{n-i}\sum_{s,w=0}^{\infty}\frac{2\alpha\lambda\left(-1\right)^{j+s+k+l}\left[\lambda\left(s+w+1\right)\right]^{k}}{k!\left(k+l+1\right)\mathrm{B}\left(i,n-i+1\right)}\binom{n-i}{j} \\ &\times\binom{\alpha\left(j+i\right)-1}{s}\binom{-\alpha\left(j+i\right)-1}{w}\binom{-k-2}{l}. \end{aligned}$$

Then, the density function of the OEHL-G order statistics is a mixture of Exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those of T_{k+l+1}.

The qth moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^{q}\right)=\sum_{k,l=0}^{\infty}b_{k,l}\;\;E\left(T_{k+l+1}\right). \tag{21}$$

Based on the moments in equation (21), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable OEHL-G order statistics. The rth L-moments is given by

$$\lambda_{r}=\frac{1}{r}\sum_{d=0}^{r-1}\left(-1\right)^{d}\binom{r-1}{d}E\left(X_{r-d:r}\right),\;r\geq 1.$$

4.7 PWMs

The PWM is the expectation of certain function of a random variable whose mean exists. A general theory for the PWMs covers the summarization and description of theoretical probability distributions and observed data samples, nonparametric estimation of the underlying distribution of an observed sample, estimation of parameters, quantiles of probability distributions and hypothesis tests. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.

The (j,i)th PWM of X following the OEHL-G distribution, say $\rho_{j,i}$, is formally defined by

$$\rho_{j,i} = E\left\{X^j F(X)^i\right\} = \int_{-\infty}^{\infty} x^j f\left(x\right) F(X)^i \,\mathrm{d}x.$$

From equation (20), we can write

$$f\left(x\right) F\left(x\right)^i = 2\alpha\lambda g(x) \sum_{s,w,k,l=0}^{\infty} \frac{(-1)^{s+k+l}}{k!} \left[\lambda\left(s+w+1\right)\right]^k \\ \times \binom{\alpha\left(i+1\right)-1}{s} \binom{-\alpha\left(i+1\right)-1}{w} \binom{-k-2}{l} G(x)^{k+l}.$$

The last equation can be expressed as

$$f\left(x\right) F(X)^i = \sum_{k,l=0}^{\infty} m_{k,l}\; h_{k+l+1}\left(x\right),$$

where

$$m_{k,l} = 2\alpha\lambda \sum_{s,w=0}^{\infty} \frac{(-1)^{s+k+l} \left[\lambda\left(s+w+1\right)\right]^k}{k!\left(k+l+1\right)} \\ \times \binom{\alpha\left(i+1\right)-1}{s} \binom{-\alpha\left(i+1\right)-1}{w} \binom{-k-2}{l}.$$

Then, the PWM of X is given by

$$\rho_{j,i} = \sum_{k,l=0}^{\infty} m_{k,l} \int_{-\infty}^{\infty} x^j\; h_{k+l+1}\left(x\right) \mathrm{d}x = \sum_{k,l=0}^{\infty} m_{k,l}\; E\left(T_{k+l+1}^j\right).$$

5. Characterizations

This section deals with certain characterizations of OEHL-G distribution. These characterizations are based on: (i) a simple relation between two truncated moments; (ii) the hazard function ; (iii) the reverse hazard function and (iv) conditional expectation of a function of the random variable. One of the advantages of characterization (i) is that the cdf is not required to have a closed form. We present our characterizations $(i) - (iv)$ in four subsections.

5.1 Characterizations based on ratio of two truncated moments

In this subsection we present characterizations of OEHL-G distribution in terms of a simple relationship between two truncated moments. This characterization result employs a theorem due to Glänzel (1987), see Theorem A.1 of Appendix A. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could also be applied when the cdf F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Proposition 5.1 Let $X:\Omega\to\mathrm{R}$ be a continuous random variable and let $q_1(x)=\left[1+\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha+1}$ and $q_2(x)=q_1(x)\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}$ for $x\in\mathrm{R}$. The random variable X has pdf (2) if and only if the function η defined in Theorem A.1 has the form

$$\eta(x)=\frac{1}{2}\left\{1+\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\},\quad x\in\mathrm{R}.$$

Proof Let X be a random variable with pdf (2), then

$$(1-F(x))\,E\left[q_1(x)\mid X\geq x\right]=2\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\},\quad x\in\mathrm{R},$$

and

$$(1-F(x))\,E\left[q_2(x)\mid X\geq x\right]=\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{2\alpha}\right\},\quad x\in\mathrm{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{1}{2}q_1(x)\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\}>0\quad for\quad x\in\mathrm{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\lambda g(x;\xi)\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha-1}}{\overline{G}(x;\xi)^2\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\}},\ x\in\mathrm{R},$$

and hence

$$s(x)=-\log\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\},\quad x\in\mathrm{R}.$$

Now, in view of Theorem A.1, X has density (2).

Corollary 5.2 Let $X:\Omega\to\mathrm{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5.1. The pdf of X is (2) if and only if there exist functions q_2 and η defined in Theorem A.1 satisfying the differential equation

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=\frac{\alpha\lambda g(x;\xi)\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha-1}}{\overline{G}(x;\xi)^2\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\}},\ x\in\mathrm{R}.$$

The general solution of the differential equation in Corollary 5.2 is

$$\eta(x)=\left\{1-\left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha}\right\}^{-1}\left[\begin{array}{c}-\int\alpha\lambda g(x;\xi)\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\times\\ \left[1-\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{\alpha-1}(q_1(x))^{-1}q_2(x)+D\end{array}\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 5.1 with $D=\frac{1}{2}$. However, it should be also noted that there are other triplets (q_1,q_2,η) satisfying the conditions of Theorem A.1.

5.2 Characterization based on hazard function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establish a non-trivial characterization of OELH-G distribution, when $\alpha=1$, which is not of the above trivial form.

Proposition 5.3 Let $X:\Omega\to\mathrm{R}$ be a continuous random variable. The pdf of X for $\alpha=1$, is (2) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h_F'(x)-\frac{g'(x;\xi)}{g(x;\xi)}h_F(x)=\lambda g(x;\xi)\frac{d}{dx}\left\{\overline{G}(x;\xi)^{-2}\left[1+\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{-1}\right\},\quad x\in\mathrm{R}.$$

Proof If X has pdf (2), then clearly the above differential equation holds. Now, this differential equation holds, then

$$\frac{d}{dx}\left\{g(x;\xi)^{-1}h_F(x)\right\}=\lambda\frac{d}{dx}\left\{\overline{G}(x;\xi)^{-2}\left[1+\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]^{-1}\right\},\quad x\in\mathrm{R}.$$

from which, we obtain

$$h_F(x)=\frac{\lambda g(x;\xi)}{\overline{G}(x;\xi)^2\left[1+\exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]},\quad x\in\mathrm{R},$$

which is the hazard function of OELH-G distribution for $\alpha=1$.

5.3 Characterizations in terms of the reverse hazard function

The reverse hazard function, r_F, of a twice differentiable distribution function, F , is defined as

$$r_F(x) = \frac{f(x)}{F(x)}, \quad x \in support\ of\ F.$$

This subsection deals with the characterizations of OEHL-G distribution based on the reverse hazard function.

Proposition 5.4 Let $X : \Omega \to \mathrm{R}$ be a continuous random variable. The random variable X has pdf (2) if and only if its reverse hazard function $r_F(x)$ satisfies the following differential equation

$$r_F'(x) - \frac{g'(x;\xi)}{g(x;\xi)} r_F(x) = 2\alpha\lambda g(x;\xi) \frac{d}{dx} \left\{ \overline{G}(x;\xi)^{-2} \times \left[\exp\left(\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right) - \exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right) \right]^{-1} \right\}, \quad x \in \mathrm{R}.$$

Proof If X has pdf (2) , then clearly the above differential equation holds. If this differential equation holds, then

$$\frac{d}{dx} \left\{ g(x;\xi)^{-1} r_F(x) \right\} = 2\alpha\lambda \frac{d}{dx} \left\{ \overline{G}(x;\xi)^{-2} \left[\exp\left(\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right) - \exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right) \right]^{-1} \right\},$$

from which, we have

$$r_F(x) = \frac{2\alpha\lambda g(x;\xi) \exp\left(-\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)}{\overline{G}(x;\xi)^2 \left[1 - \exp\left(-2\frac{\lambda G(x;\xi)}{\overline{G}(x;\xi)}\right)\right]}, \quad x \in \mathrm{R}.$$

5.4 Characterization based on the conditional expectation of certain functions of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it as a proposition here, which can be used to characterize OELH-G distribution.

Proposition 5.5 Let $X : \Omega \to (d, e)$ be a continuous random variable with cdf F . Let $\psi(x)$ be a differentiable function on (d, e) with $\lim_{x \to e^-} \psi(x) = 1$. Then, for $\delta \neq 1$,

$$E[\psi(X) \mid X \leq x] = \delta\psi(x), \quad x \in (d, e)$$

implies

$$\psi(x) = (F(x))^{\frac{1}{\delta} - 1}, \quad x \in (d, e).$$

REMARK 5.1 For $(d, e) = \mathrm{R}$,

$$\psi(x) = \frac{1 - \exp\left(-\frac{\lambda G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right)}{1 + \exp\left(-\frac{\lambda G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right)}$$

and $\delta = \frac{\alpha}{\alpha+1}$, Proposition 5.5 provides a characterization of OELH-G distribution.

6. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimators (MLEs) enjoy desirable properties and can be used to obtain confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood method.

Let $X_1, ..., X_n$ be a random sample from the OEHL-G family with parameters α, λ and $\boldsymbol{\xi}$. Let $\theta = (\alpha, \lambda, \boldsymbol{\xi}^\intercal)^\intercal$ be the $p \times 1$ parameter vector. To obtain the MLE of θ, the log-likelihood function is given by

$$\begin{aligned}
\ell(\theta) = n\log(2\alpha) &+ \sum_{i=1}^{n} \log g(x_i;\boldsymbol{\xi}) - 2\sum_{i=1}^{n} \log\left[\bar{G}(x_i;\boldsymbol{\xi})\right] \\
&-\lambda \sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})} + (\alpha-1)\sum_{i=1}^{n} \log\left\{1 - \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\} \\
&+n\log(\lambda) - (\alpha+1)\sum_{i=1}^{n} \log\left\{1 + \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\}.
\end{aligned}$$

Then, the score vector components, $\mathbf{U}(\theta) = \frac{\partial \ell}{\partial \theta} = \left(U_\alpha, U_\lambda, U_{\boldsymbol{\xi}_k}\right)^\intercal$, are

$$U_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left\{1 - \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\} - \sum_{i=1}^{n} \log\left\{1 + \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\},$$

$$\begin{aligned}
U_\lambda = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})} &+ (\alpha-1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})\exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]}{\bar{G}(x_i;\boldsymbol{\xi})\left\{1 - \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\}} \\
&+ (\alpha+1)\sum_{i=1}^{n} \frac{G(x_i;\boldsymbol{\xi})\exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]}{\bar{G}(x_i;\boldsymbol{\xi})\left\{1 + \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\}}
\end{aligned}$$

and

$$\begin{aligned}
U_{\boldsymbol{\xi}_k} &= \sum_{i=1}^{n} \frac{g'(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})} - 2\sum_{i=1}^{n} \frac{\bar{G}'(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})} \\
&\quad -\lambda \sum_{i=1}^{n} \frac{\bar{G}(x_i;\boldsymbol{\xi})G'(x_i;\boldsymbol{\xi}) - G(x_i;\boldsymbol{\xi})\bar{G}'(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})^2} \\
&\quad +\lambda(\alpha-1) \sum_{i=1}^{n} \frac{\left[\bar{G}(x_i;\boldsymbol{\xi})G'(x_i;\boldsymbol{\xi}) - G(x_i;\boldsymbol{\xi})\bar{G}'(x_i;\boldsymbol{\xi})\right] \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]}{\bar{G}(x_i;\boldsymbol{\xi})^2 \left\{1 - \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\}} \\
&\quad +\lambda(\alpha+1) \sum_{i=1}^{n} \frac{\left[\bar{G}(x_i;\boldsymbol{\xi})G'(x_i;\boldsymbol{\xi}) - G(x_i;\boldsymbol{\xi})\bar{G}'(x_i;\boldsymbol{\xi})\right] \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]}{\bar{G}(x_i;\boldsymbol{\xi})^2 \left\{1 + \exp\left[\frac{-\lambda G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]\right\}},
\end{aligned}$$

where $g'(x_i;\boldsymbol{\xi}) = \partial g(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k$, $G'(x_i;\boldsymbol{\xi}) = \partial G(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k$ and $\bar{G}'(x_i;\boldsymbol{\xi}) = \partial\bar{G}(x_i;\boldsymbol{\xi})/\partial\boldsymbol{\xi}_k$.

Setting the nonlinear system of equations $U_\alpha = U_\lambda = 0$ and $U_{\boldsymbol{\xi}_\mathbf{k}} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\theta} = (\widehat{\alpha}, \widehat{\lambda}, \widehat{\boldsymbol{\xi}}^\intercal)^\intercal$. To do this, it is usually more convenient to adopt nonlinear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\theta) = \{\frac{\partial^2\ell}{\partial r\,\partial s}\}$ (for $r, s = \alpha, \lambda, \boldsymbol{\xi}$), whose elements are given in appendix A and can be computed numerically.

Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\theta})^{-1})$ distribution to obtain confidence intervals for the parameters. Here, $J(\widehat{\theta})$ is the total observed information matrix evaluated at $\widehat{\theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. Improved MLEs can be obtained for the new family using second-order bias corrections. However, these corrected estimates depend on cumulants of log-likelihood derivatives and will be addressed in future research.

7. Applications

In this section, we provide applications to three real data sets to illustrate the importance of the OEHL-W and OEHL-Ga distributions presented in Section 2. The goodness-of-fit statistics for these distributions and other competitive distributions are compared and the MLEs of their parameters are provided. In order to compare the fitted distributions, we consider goodness-of-fit measures including $-\widehat{\ell}$, Anderson-Darling statistic (A^*) and Cramér-von Mises statistic (W^*), where $\widehat{\ell}$ denotes the maximized log-likelihood, Generally, the smaller these statistics are, the better the fit.

7.1 Relief times of twenty patients

The first real data set is taken from Gross and Clark (1975, p.105), which gives the relief times of 20 patients receiving an analgesic. The data are as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. We compare the fits of the OEHL-W distribution with other competitive distributions, namely: the Weibull (W), odd log-logistic Weibull (OLL-W), generalized odd log-logistic Weibull (GOLL-W), Kumaraswamy Weibull (Kum-W), exponentiated generalized Weibull (EG-W) and exponentiated half-logistic Weibull (EHL-W) distributions. The MLEs of the model parameters, their corresponding standard errors (in parentheses) and the values of $-\widehat{\ell}$, A^* and W^* are given in Table 1. Table 1 compares the fits of the OEHL-W distribution with the EHL-W, EG-W, Kum-W, GOLL-W, OLL-W and W distributions. The results in Table 1 show that the OEHL-W distribution has the lowest values for the $-\widehat{\ell}$, A^* and W^* statistics among the fitted models. So, the OEHL-W distribution could be chosen as the best model.

Table 1. MLEs, their standard errors and goodness-of-fit statistics for the relief times data.

Model	α	λ	a	b	$-\widehat{\ell}$	A^*	W^*
W			2.787	2.130	20.586	1.092	0.185
			(0.427)	(0.182)			
OLL-W	47.157		0.09	103	16.525	0.318	0.053
	(19.915)		(0.034)	(159.847)			
GOLL-W	528.735	1.483	1.691	0.58	16.479	0.310	0.051
	(0.32)	(0.003)	(0.0008)	(0.368)			
Kum-W	1.047	0.146	2.422	0.93	20.477	0.994	0.168
	(0.565)	(0.034)	(0.002)	(0.002)			
EG-W	0.19	4.442	1.566	0.413	17.486	0.609	0.102
	(0.041)	(1.769)	(0.002)	(0.002)			
EHL-W	4.857	0.213	1.304	0.261	17.113	0.541	0.091
	(1.986)	(0.038)	(0.002)	(0.002)			
OEHL-W	$4.452 \cdot 10^6$	549.547	0.242	$3.651 \cdot 10^6$	15.414	0.165	0.029
	(4.750)	(0.121)	(0.0009)	(5.295)			

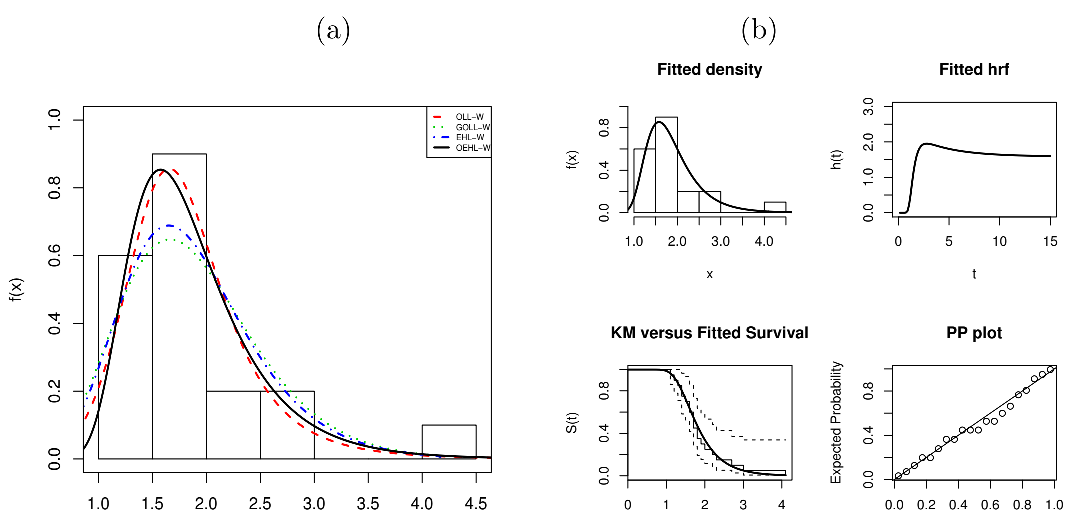

Figure 7. (a) Estimated densities of the best four models and (b) fitted functions of the OEHL-W model for strengths of glass fibers data set.

7.2 Time to failure ($10^3 h$) of turbocharger

The second data set ($n = 40$) below is from Xu et al. (2003) and it represents the time to failure ($10^3 h$) of turbocharger of one type of engine. The data are: 1.6, 3.5, 4.8, 5.4, 6.0, 6.5, 7.0, 7.3, 7.7, 8.0, 8.4, 2.0, 3.9, 5.0, 5.6, 6.1, 6.5, 7.1, 7.3, 7.8, 8.1, 8.4, 2.6, 4.5, 5.1, 5.8, 6.3, 6.7, 7.3, 7.7, 7.9, 8.3, 8.5, 3.0, 4.6, 5.3, 6.0, 8.7, 8.8, 9.0. For these data, we compare the fits of the OEHL-Ga distribution with the gamma (Ga), odd log-logistic gamma (OLL-Ga), generalized odd log-logistic gamma (GOLL-Ga), Kumaraswamy gamma (Kum-Ga), exponentiated generalized gamma (EG-Ga) and exponentiated half-logistic gamma (EHL-Ga) distributions.

Table 2 lists the MLEs of the model parameters, their corresponding standard errors (in parentheses) and the values of $-\widehat{\ell}$, A^{*} and W^{*}. Table 2 compares the fits of the OEHL-Ga distribution with the EHL-Ga, EG-Ga, Kum-Ga, GOLL-Ga, OLL-Ga and Ga distributions. The OEHL-Ga distribution has the lowest values for goodness-of-fit statistics among all fitted models. So, the OEHL-Ga distribution can be chosen as the best model.

Table 2. MLEs, their standard errors and goodness-of-fit statistics for time to failure data.

Model	α	λ	a	b	$-\widehat{\ell}$	A^{*}	W^{*}
Ga			7.718	1.234	87.410	1.361	0.205
			(1.689)	(0.279)			
OLL-Ga	3.114		1.271	0.153	86.639	1.119	0.165
	(2.942)		(1.888)	(0.298)			
GOLL-Ga	0.3007	0.037	589.539	76.756	80.479	0.369	0.055
	(0.139)	(0.024)	(201.293)	(25.720)			
Kum-Ga	0.735	42.666	6.173	0.245	83.721	0.77	0.107
	(1.104)	(51.630)	(9.102)	(0.301)			
EG-Ga	38.098	0.16	27.473	2.077	80.192	0.384	0.049
	(0.236)	(0.025)	(0.002)	(0.003)			
EHL-Ga	0.157	39.734	28.772	2.319	79.336	0.269	0.033
	(0.026)	(14.270)	(0.002)	(0.002)			
OEHL-Ga	0.345	0.054	14.136	2.621	78.126	0.139	0.023
	(0.189)	(0.095)	(11.466)	(1.819)			

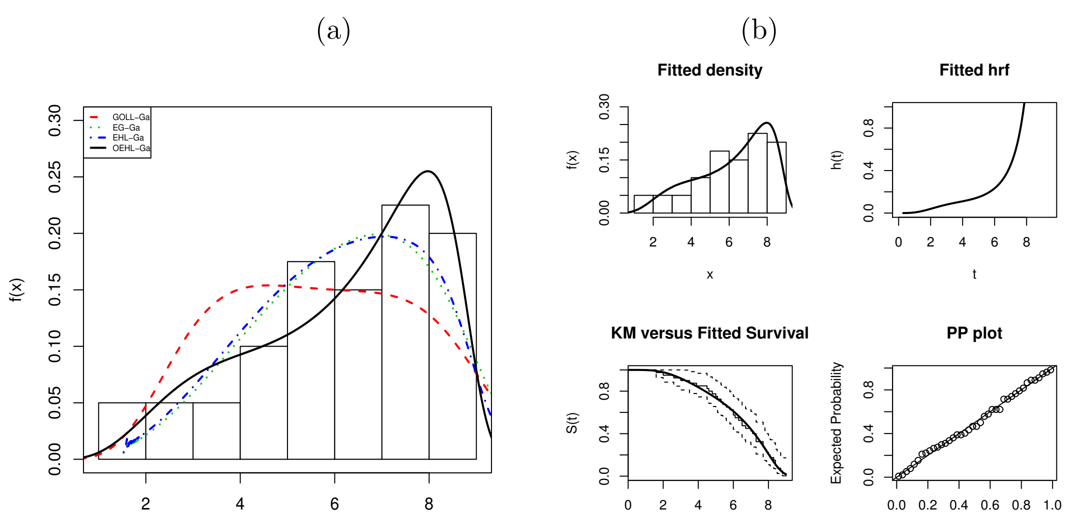

Figure 8. (a) Estimated densities of the best four models and (b) fitted functions of the OEHL-Ga model for time to failure (10^3h) of turbocharger data set.

7.3 Strengths of 1.5 cm glass fibers

The third data set ($n = 63$) consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. The data are: 0.55, 0.74, 0.77, 0.81, 0.8,4 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.3, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.5, 1.5, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.6, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.7, 1.7, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2, 2.01, 2.24.

Table 3 lists the MLEs of the model parameters, and corresponding standard errors (in parentheses) with estimated $-\widehat{\ell}$, A^* and W^* statistics. Based on the figures in Table 3, OEHL-Ga distribution provides the best fit among others.

Table 3. MLEs, their standard errors and goodness-of-fit statistics for strengths of 1.5 cm glass fibers

Models	α	λ	a	b	$A^\star$	$W^\star$	$-\widehat{\ell}$
Ga			17.437	11.572	3.117	0.568	23.952
			(3.078)	(2.072)			
OLL-Ga	5.605		1.087	0.509	2.336	0.423	20.421
	(4.794)		(1.452)	(0.925)			
GOLL-Ga	3.616	0.293	6.911	2.464	2.258	0.409	19.894
	(2.171)	(0.117)	(0.935)	(1.725)			
Kum-Ga	1.111	8.471	9.402	3.852	2.173	0.396	18.999
	(0.537)	(6.974)	(4.758)	(2.118)			
EG-Ga	43.418	0.429	18.134	5.959	1.339	0.243	15.368
	(12.492)	(0.235)	(12.454)	(6.130)			
EHL-Ga	0.797	15.302	10.527	3.926	1.390	0.254	15.238
	(0.342)	(16.298)	(5.181)	(2.813)			
OEHL-Ga	0.618	35.427	11.606	3.703	1.142	0.207	14.423
	(0.253)	(12.088)	(4.659)	(1.959)			

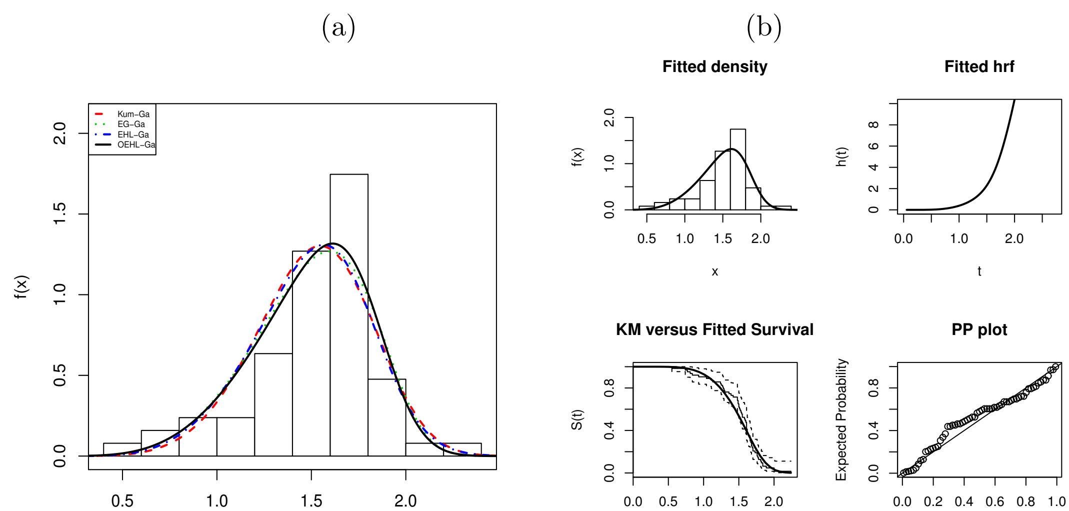

Figure 9. (a) Estimated densities of the best four models and (b) fitted functions of the OEHL-Ga model for strengths of glass fibers data set.

Moreover, Table 4 shows the Akaike Information Criteria (AIC) for all fitted models and for three data sets. Table 4 reveals that the OEHL-G family provides better fits for these three data sets than other all competitive models. It is clear from Tables 1, 2, 3 and 4 that the OEHL-W and OEHL-Ga distributions provide the best fits among others. The histograms of the fitted distributions for the OEHL-W and OEHL-Ga models are displayed in Figures 7(a), 8(a) and 9(a), respectively. Figures 7(b), 8(b) and 9(b) display the fitted pdf, estimated hrf, fitted survival functions and probability-probability (P-P) plots for the OEHL-W and OEHL-Ga models, respectively. It is evident from these plots that the OEHL-G distributions provide superior fit to the three data sets.

Table 4. AIC values of fitted models for all used data sets.

First data set		Second data set		Third data set	
Models	AIC	Models	AIC	Models	AIC
W	45.172	Ga	178.820	Ga	51.903
OLL-W	39.050	OLL-Ga	179.278	OLL-Ga	46.843
GOLL-W	40.958	GOLL-Ga	168.958	GOLL-Ga	47.788
Kum-W	48.954	Kum-Ga	175.442	Kum-Ga	45.998
EG-W	42.972	EG-Ga	168.384	EG-Ga	38.736
EHL-W	42.226	EHL-Ga	166.672	EHL-Ga	38.476
OEHL-W	38.828	OEHL-Ga	164.252	OEHL-Ga	36.846

8. Simulation study

In this section, we evaluate the performance of the maximum likelihood method for estimating the OEHL-N parameters using a Monte Carlo simulation study with $10,000$ replications. We calculate biases, the mean square errors (MSEs) of the parameter estimates, estimated average lengths (ALs) and coverage probabilities (CPs) using the R package. The MSEs, ALs and CPs can be calculated by using following equations:

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon),$$

$$\widehat{MSE}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2,$$

$$CP_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\epsilon}_i - 1.95996 s_{\hat{\epsilon}_i}, \hat{\epsilon}_i + 1.95996 s_{\hat{\epsilon}_i}),$$

$$AL_{\epsilon}(n) = \frac{3.919928}{N}\sum_{i=1}^{N} s_{\hat{\epsilon}_i}.$$

for $\epsilon = \alpha, \lambda, \mu, \sigma$.

We generate $N = 1,000$ samples of sizes $n = 50, 55, \ldots, 1000$ from the OEHL-N distribution with $\alpha = \lambda = 0.5$ and $\mu = \sigma = 2$. The numerical results for the above measures are shown in the plots of Figure 9. It is noted, from Figure 9, that the estimated biases decrease when the sample size n increases. Further, the estimated MSEs decay toward zero as n increases. This fact reveals the consistency property of the MLEs. The CPs are near to 0.95 and approach to the nominal value when the sample size increases. Moreover, if the sample size increases, the ALs decrease in each case.

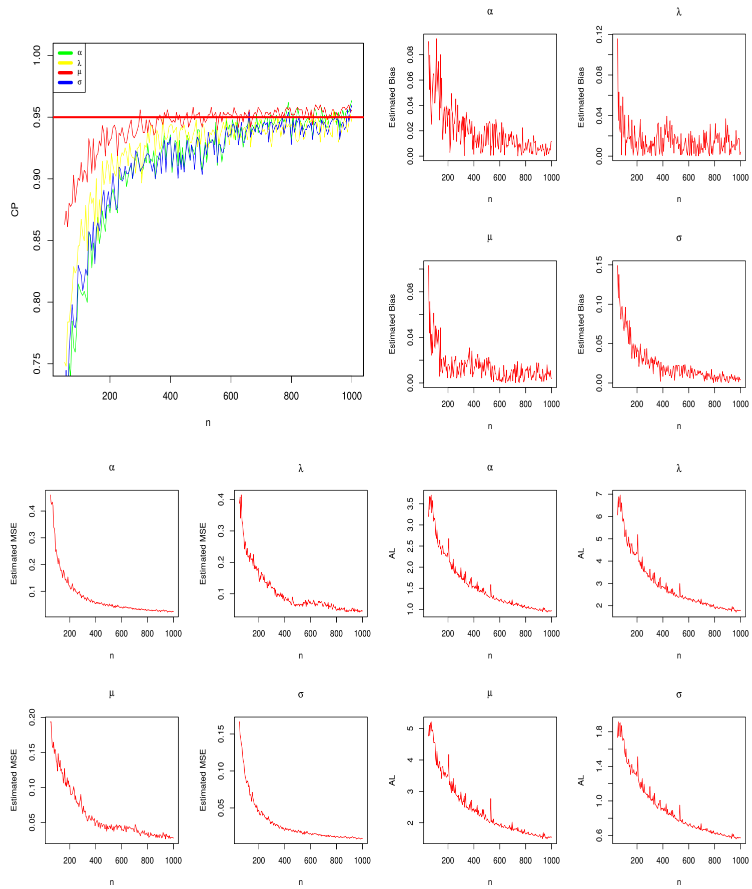

Figure 10. Estimated CPs, biases, MSEs and ALs of the selected parameter vector.

9. Conclusions

We proposed a new *odd exponentiated half-logistic-G* (OEHL-G) family of distributions with two extra shape parameters. Many well-known distributions emerge as special cases of the OEHL-G family. The mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, quantile and generating functions, entropies, order statistics and probability weighted moments have been provided. The model parameters have been estimated by the maximum likelihood estimation method. It has been shown, by means of two real data sets, that special cases of the OEHL-G family can provide better fits than other competitive models generated using well-known families. A graphical simulation to assess the performance of the maximum likelihood estimators is provided. We hope that the OEHL-G family may be extensively used in statistics.

Appendix A.

Theorem A.1 Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [a, b]$ be an interval for some $d < b$ $(a = -\infty,\ b = \infty$ might as well be allowed$)$. Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let q_1 and q_2 be two real functions defined on H such that

$$\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right]\eta(x), \quad x \in H,$$

is defined with some real function η. Assume that $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\eta q_1 = q_2$ has no real solution in the interior of H. Then F is uniquely determined by the functions q_1, q_2 and η , particularly

$$F(x) = \int_a^x C\left|\frac{\eta'(u)}{\eta(u)\, q_1(u) - q_2(u)}\right| \exp(-s(u))\, \mathrm{d}u,$$

where the function s is a solution of the differential equation $s' = \frac{\eta'\, q_1}{\eta q_1 - q_2}$ and C is the normalization constant, such that $\int_H \mathrm{d}F = 1$.

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